Transmission regularity: Behavior and preservation under graph products

Yaser Alizadeh[a], Sandi Klavžar[b,c,d], Fateme Nikfetrat[e]

[a,e] Department of Mathematics, Hakim Sabzevari University, Sabzevar, Iran

y.alizadeh@hsu.ac.ir

fateme1380nik@gmail.com

[b] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

sandi.klavzar@fmf.uni-lj.si

[c] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

[d] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

Abstract

The transmission of a vertex $v \in V(G)$ in a connected graph G is the sum of the distances from v to all other vertices of G. A graph is transmission regular if all of its vertices have the same transmission value. In this paper, the behavior of transmission regularity is investigated under corona product, lexicographic product, and disjunction. The existence problem of transmission-regular graphs with prescribed cyclomatic number is also addressed.

Keywords: transmission; transmission regular graph; graph product; Wiener complexity; cyclomatic number

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1 Introduction

All considered graphs are connected and simple. Let $G = (V(G), E(G))$ be a graph. The order and the size of a graph G are denoted by $n(G)$ and $m(G)$, respectively. The distance $d_G(u, v)$ (briefly $d(u, v)$) between vertices $u, v \in V(G)$ is the length of a shortest path connecting them. The diameter $\mathrm{diam}(G)$ of G is the maximum distance

among pair of its vertices. The degree of $u \in V(G)$ is denoted by $\deg_G(u)$ (briefly $\deg(u)$). The *degree complexity* $C_d(G)$ of G is defined as

$$C_d(G) = |\{\deg_G(u) : \ u \in V(G)\}|.$$

The *transmission* $\sigma_G(v)$ of $v \in V(G)$ is

$$\sigma_G(v) = \sum_{u \in V(G)} d(v,u)\,,$$

and the *Wiener complexity* [3] $C_{\rm W}(G)$ of G is the number of distinct transmission values among all vertices of G, that is,

$$C_{\rm W}(G) = |\{\sigma_G(v) : \ u \in V(G)\}|\,.$$

Transmission plays a fundamental role in the construction of many distance-based invariants and topological indices for both abstract and molecular graphs, cf. [13, 14, 24, 25]. The Wiener complexity serves as a measure of the structural diversity of the graph based on the distribution of vertex transmissions. The earliest studies focusing on transmission were conducted in [20, 21].

More recently, several graph families have been defined based on the transmission properties of their vertices. An important class of such graphs consists of *transmission irregular graphs* (*TI graphs*, for short) defined as the graphs in which every vertex has a distinct transmission value. Studies on TI graphs have been presented in [3, 9, 12]. In [25] new families of chemical graphs exhibiting transmission irregularity were introduced, and necessary conditions for the uniqueness of vertex transmission values within these molecular structures were established.

A graph is termed *stepwise transmission irregular* (*STI* for short), whenever $|\sigma_G(u) - \sigma_G(v)| = 1$ holds for all edges $uv \in E(G)$. Fundamental properties of STI graphs, along with the construction of several infinite families are presented in [4, 10, 15]. in [5], STI graphs were generalized to k-STI graphs, which are the graphs in which the absolute difference between the transmission values of every two distinct vertices is equal to k.

On the other side, however, there are transmission-regular graphs [17] is one in which all vertices have the same transmission. Such graphs are referred to as distance-balanced graphs in [2], we shortly call them *TR-graphs*. In [1], it was shown that every TR-graph is 2-connected. The authors further proved that every vertex-transitive graph is transmission regular. Moreover, they constructed an infinite family of TR-graphs that are not vertex-transitive. In [19], TR-graphs were studied under the notion of distance-balanced graphs and several general results on ℓ-distance-balanced graphs were established, together with a variety of illustrative examples.

In this paper, we investigate the preservation of transmission regularity under under the corona product, the lexicographic product, and the disjunction. Moreover, we derive sharp bounds for the Wiener complexity of composite graphs and study the existence of transmission-regular graphs with prescribed cyclomatic number.

2 TR property under graph operations

Our goal in this section is to consider different graph operations and to characterize those of them that preserve transmission regularity. In particular, we identify conditions under which the resulting graphs are TR. A particular result of this kind was proved in [22, Corollary of Theorem 2] asserting that the join of graphs G and H is TR if and only if G and H are r-regular and k-regular, respectively, such that $n(G) - r = n(H) - k$.

Results obtained here provide criteria for determining transmission regularity and illustrate how different graph operations affect this property. Each of the following subsections covers a single operation.

2.1 Corona products

Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$, and let H be a graph. Then the *corona product* $G \circ H$ of G and H is the graph obtained from the disjoint union of G and n (disjoint) copies of H, say $H_1, \ldots H_n$, where v_i, $i \in [n]$, is adjacent to all the vertices of H_i.

In [22, Corollary of Theorem 3] is proved that the corona product of two arbitrary, nontrivial and connected graphs is not TR. This immediately implies that $G \circ H$ is TR if and only if $n(G) = 1$ and H is a complete graph. We next present the following extension of this result.

Theorem 2.1 *If G and H are graphs, then*

$$C_{\mathrm{W}}(G) \leq C_{\mathrm{W}}(G \circ H) \leq C_{\mathrm{W}}(G)(C_{\mathrm{d}}(H) + 1).$$

Moreover, the left equality holds if and only if $n(G) = 1$ and H is a complete graph, and the right inequality is sharp.

Proof. For every vertex $x \in V(G)$ we have

$$\begin{aligned}\sigma_{G\circ H}(x) &= n(H) + \sum_{y \in V(G)\setminus\{x\}} \left(d_G(x,y) + \sigma_G(x)\right) n(H) + \sigma_G(x) \\ &= n(H) + n(H)(\sigma_G(x) + n(G) - 1) \\ &= (n(H) + 1)\sigma_G(x) + n(H)n(G)\,.\end{aligned}$$

Hence distinct transmission values of the vertices of G yield distinct transmission values in $G \circ H$. Consequently, $C_{\rm W}(G) \le C_{\rm W}(G \circ H)$.

We next consider the equality case in the just proved lower bound. For this sake we infer that for each vertex $z \in H_x$, where H_x is the isomorphic copy of H to all of which vertices x is adjacent, the equality

$$\sigma_{G\circ H}(z) - \sigma_{G\circ H}(x) = n(G \circ H) - \deg_{H_x}(z) - 2 \tag{1}$$

implies that

$$\sigma_{G\circ H}(z) = \sigma_{G\circ H}(x) + n(G \circ H) - \deg_H(z) - 2 \ge 0\,. \tag{2}$$

Now, the equality in $C_{\rm W}(G) \le C_{\rm W}(G \circ H)$ holds if and only if $\sigma_{G\circ H}(z) = \sigma_{G\circ H}(x)$ for each $x \in V(G)$ and $z \in V(H_x)$. In this case, (1) gives us

$$n(G \circ H) = \deg_{H_x}(z) + 2 \le n(H) + 1\,,$$

which in turn implies that

$$n(G \circ H) = n(G)(n(H) + 1) \le n(H) + 1 \le n(G \circ H)\,.$$

Consequently, the equality in the lower bound holds if and only if $n(G) = 1$ and $\deg_{H_x}(z) = n(H) - 1$, for each $z \in H_x$.

For the upper bound just observe that for any two vertices $u, v \in V(H_x)$, we have $\deg_H(u) = \deg_H(v)$ if and only if $\sigma_{\rm G\circ H}(u) = \sigma_{\rm G\circ H}(v)$. From this it follows that $C_{\rm W}(G \circ H) \le C_{\rm W}(G)\,(C_{\rm d}(H) + 1)$.

Finally, the sharpness of the upper bound can be demonstrated by an arbitrary TR-graph G. □

The lower bound of Theorem 2.1 can be improved as soon as the base graph G is of order at least two.

Proposition 2.2 *If G is a graph with $n(G) \ge 2$, then $C_{\rm W}(G \circ H) \ge C_{\rm W}(G) + C_{\rm d}(H)$.*

Proof. Let $C_{\rm W}(G) = k$, and select vertices $v_1, \ldots, v_k$ of G such that

$$\sigma_G(v_1) < \cdots < \sigma_G(v_k).$$

Similarly, let $C_{\rm d}(H) = t$, and select vertices $u_1, \ldots, u_t$ of H such that

$$\deg_H(u_1) < \cdots < \deg_H(u_t).$$

Since $n(G) \ge 2$, for every $x \in V(G)$ and every $u \in V(H_x)$, we have

$$\sigma_{\rm G\circ H}(u) > \sigma_{\rm G\circ H}(x).$$

By this and (1) and (2), we obtain the chain

$$\sigma_{\rm G\circ H}(v_1) < \sigma_{\rm G\circ H}(v_2) < \cdots < \sigma_{\rm G\circ H}(v_k) < \sigma_{\rm G\circ H}(u_t) < \cdots < \sigma_{\rm G\circ H}(u_1)\,,$$

where, for each $i \in [t]$, the vertex u_i is considered to lie in $V(H_{v_k})$. □

2.2 Lexicographic products

The *lexicographic product* $G[H]$ of graphs G and H is the graph with the vertex set $V(G[H]) = V(G) \times V(H)$. Two vertices (u, v) and (u', v') are adjacent in $G[H]$ if either $uu' \in E(G)$, or $u = u'$ and $vv' \in E(H)$. For a thorough discussion of this product, see the book [16]. In particular, the following distance formula holds (cf. also [7]).

Lemma 2.3 *If (u, v) and (x, y) are two vertices of $G[H]$, then*

$$d_{G[H]}((u,v),(x,y)) = \begin{cases} d_G(u,x) & \text{if } u \neq x \\ 1 & \text{if } u = x, vy \in E(H) \\ 2 & \text{if } u = x, vy \notin E(H) \end{cases}$$

Using this distance formula, the following can be derived.

Proposition 2.4 *Let G and H be graphs and $(u, v) \in V(G[H])$, then*

$$\sigma_{G[H]}((u,v)) = n(H)\sigma_G(u) - deg_H(v) + 2(n(H) - 1) \tag{3}$$

Proof. By Lemma 2.3, the transmission of each vertex $(u, v) \in V(G[H])$ can be determined as follows:

$$\begin{aligned} \sigma_{G[H]}((u,v)) &= \sum_{(x,y)\in V(G[H])} d_{G[H]}((u,v),(x,y)) \\ &= \sum_{y\in V(H)} \sum_{x\in V(G)} d_G(u,x) + \sum_{y\in V(H), vy\in E(H)} d_{G[H]}((u,v),(u,y)) \\ &+ \sum_{y\in V(H), vy\notin E(H)} d_{G[H]}((u,v),(u,y)) \\ &= n(H)\sigma_G(u) + deg_H(v) + 2(n(H) - deg_H(v) - 1). \end{aligned}$$

Therefore, the result holds. □

The above proposition leads to the following characterization.

Theorem 2.5 *The lexicographic product $G[H]$ of graphs G and H is a TR-graph if and only if G is a TR-graph and H is a regular graph.*

Proof. According to Proposition 2.4, since the transmission of each vertex (u, v) in the lexicographic product $G[H]$ depends on the $\sigma_G(u)$ and $deg_H(v)$, the graph is transmission-regular if and only if G is a TR-graph and H is a regular graph. □

In general, the Wiener complexity of lexicographic products can be bounded as follows.

Proposition 2.6 *If G and H are graphs, then*

$$\max\{C_{\rm W}(G), C_{\rm d}(H)\} \le C_{\rm W}(G[H]) \le C_{\rm W}(G)C_{\rm d}(H)\,.$$

Proof. By Proposition 2.4, the lower bound follows by fixing the transmission values of one of the factor graphs. The upper bound is straightforward. □

2.3 Disjunctive products

The *disjunction* $G \vee H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$, and vertices (u, v) and (x, y) are adjacent whenever $ux \in E(G)$ or $vy \in E(H)$, cf. [8]. The disjunctive product is also known as the co-normal product, see [6, 23].

The following result follows immediately by the inclusion-exclusion principle.

Lemma 2.7 *If $(u, v) \in V(G \vee H)$, then*

$$\deg_{G\vee H}((u, v)) = n(H)\deg_G(u) + n(G)\deg_H(v) - \deg_G(u)\deg_H(v)\,.$$

We now have a parallel result to the one for TR lexicographic products, that is, to Theorem 2.5.

Theorem 2.8 *If $n(G) \ge 2$ and $n(H) \ge 2$, then $G \vee H$ is a TR-graph if G and H both are regular graphs.*

Proof. Let $uu' \in E(G)$ and $yy' \in E(H)$. Then, by definition, for every pair of vertices $(u, v), (x, y) \in V(G \vee H)$, there is a common neighbor $(u', y') \in V(G \vee H)$. So $\operatorname{diam}(G \vee H) \le 2$. If $\operatorname{diam}(G \vee H) = 1$, that is, if $G \vee H$ is a complete graph, then both G and H are complete graphs, and therefore they are regular. Assume next that $\operatorname{diam}(G \vee H) = 2$. In this case, for each pair of vertices $(u, v), (u, v') \in V(G \vee H)$ we have

$$|\sigma_{G\vee H}((u, v)) - \sigma_{G\vee H}((u, v'))| = |\deg_{G\vee H}((u, v)) - \deg_{G\vee H}((u, v'))| = 0\,,$$

so $\deg_{G\vee H}((u, v)) = \deg_{G\vee H}((u, v'))$. Using Lemma 2.7 we hence deduce that

$$\begin{aligned}\deg_{G\vee H}((u, v)) &= n(H)\deg_G(u) + n(G)\deg_H(v) - \deg_G(u)\deg_H(v),\\ \deg_{G\vee H}((u, v')) &= n(H)\deg_G(u) + n(G)\deg_H(v') - \deg_G(u)\deg_H(v')\,.\end{aligned}$$

Therefore, $(n(G) - \deg_G(u))\deg_H(v') = (n(G) - \deg_G(u))\deg_H(v)$, and so $\deg_H(v) = \deg_H(v')$. Thus H is a regular graph. Similarly it can be proven for any pair of $u, u' \in V(G)$ we have $\deg_G(u) = \deg_G(u')$, that is, G is a regular graph too. □

The Wiener complexity itself of disjunctive products can be bounded as follows.

Proposition 2.9 *If G and H are graphs, then*

$$\max\{C_{\rm d}(G), C_{\rm d}(H)\} \leq C_{\rm W}(G \vee H) \leq C_{\rm d}(G)C_{\rm d}(H)\,.$$

Moreover, both bounds are sharp.

Proof. Te already established fact that $\mathrm{diam}(G \vee H) \leq 2$ implies that $C_{\rm W}(G \vee H) = C_{\rm d}(G \vee H)$. By Lemma 2.7 it thus follows that

$$max\{C_{\rm d}(G), C_{\rm d}(H)\} \leq C_{\rm d}(G \vee H) \leq C_{\rm d}(G)C_{\rm d}(H).$$

The bounds are sharp and equalities hold when G or H are regular. □

3 TR-graphs with given cyclomatic number

In this section, we investigate the existence of transmission regular graphs with prescribed cyclomatic numbers. Recall that the *cyclomatic number* $c(G)$ of G is defined by $c(G) = m - n + 1$. Our initial motivation for this section is the following research result.

Proposition 3.1 [1, Proposition 4.3] *If G is a TR-graph of order at least 3, then G is 2-connected.*

It is worth noting the following direct consequence of Proposition 3.1.

Corollary 3.2 *If G is a unicyclic graph, then G is a TR-graph if and only if G is a cycle graph.*

Let G be a graph and let $uv \in E(G)$. Then by n_u we denote the number of vertices that are closer to u than to v. In addition, for a subset $S \subseteq V(G)$, we will denote by $n_v(S)$ the number of vertices in S that are closer to v than to u. From our perspective, the following result, which first appears (implicitly) in [11], is useful.

Lemma 3.3 *If $uv \in E(G)$, then $\sigma_G(u) - \sigma_G(v) = n_v - n_u$.*

While among the unicyclic graphs we find at least cycles to be TR-graphs, there are no such graphs among the graphs of cyclomatic number 2.

Proposition 3.4 *There exists no TR-graph of cyclomatic number 2.*

Proof. Suppose on the contrary that G is a TR-graph with $c(G) = 2$. By Proposition 3.1 we infer that G is isomorphic to a graph obtained from a cycle by adding exactly one chord. Let uv be the chord of G and let $x \neq u$ be a vertex adjacent to v. Consider the cycle C of G containing the vertices v and x. Then it is straightforward to verify that $n_v(V(C)) = n_x(V(C))$. On the other hand, this equality does not hold for the vertices from $V(G) \setminus V(C)$. Hence, $n_v(V(G) \setminus V(C)) > n_x(V(G) \setminus V(C))$, which in view of Lemma 3.3 contradicts the transmission regularity of G. □

Proposition 3.5 *If G is a TR-graph with $c(G) = 3$, then $G \cong K_4$ or G contains a vertex of degree 2.*

Proof. Let G be a transmission-regular graph with $c(G) = 3$. Assume that that every vertex of G has degree at least 3. Then the handshaking lemma yields $2m(G) \geq 3n(G)$. Since $c(G) = 3$, we also have $m(G) = n(G) + 2$. These two facts imply $n(G) \leq 4$. Since among all graphs with $n(G) \leq 4$, the complete graph K_4 is the unique graph whose $c(G) = 3$, the conclusion follows. □

In connection with Proposition 3.5 it seems to us that the only TR-graph of cyclomatic number 3 is K_4.

To construct TR-graphs with larger cyclomatic number, the Cartesian product of graphs is useful. Recall that the *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph whose vertex set is $V(G) \times V(H)$, and vertices (u, v) and (x, y) in $G \,\square\, H$ are adjacent if either $u = x$ and $vy \in E(H)$, or $v = y$ and $ux \in E(G)$. We need the following result.

Theorem 3.6 [18, Proposition 4.1] *The Cartesian product $G \,\square\, H$ is TR if and only if both G and H are TR-graphs.*

Proposition 3.7 *For every integer $k \geq 4$, there exists a TR-graph of cyclomatic number k.*

Proof. Consider the family of graphs $C_{k-1} \,\square\, K_2$, $k \geq 4$. Since every cycle is transmission regular and K_2 is also transmission regular, it follows from Theorem 3.6, that $C_{k-1} \,\square\, K_2$ is transmission regular with $c(C_{k-1} \,\square\, K_2) = 3(k-1) - 2(k-1) + 1 = k$. □

4 Concluding remarks

This work is concerned with two main problems: the preservation of transmission regularity under various fundamental graph operations, and the existence of transmission-regular graphs with prescribed cyclomatic numbers. The results obtained highlight the influence of structural parameters such as degree distribution, connectivity, and cycle rank on transmission regularity. In particular, constructing an infinite family of transmission-regular graphs with prescribed cyclomatic number $k \geq 3$, remains an open problem. Future research directions include a deeper structural analysis of transmission-regular graphs, focusing on extremal configurations, symmetry properties, and decomposition techniques. Another promising line of study concerns topological indices derived from transmission, particularly in chemical graph theory. Moreover, exploring algorithmic aspects of detecting transmission regularity and examining its interaction with other distance-based graph invariants may yield further insights.

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Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflict of Interest Statement

The authors declare no conflict of interest.

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